F. anal. 3.3. 2025

- Slides, info (?)

A. F.-transform in $C_m(\mathbb{R})$ and $L^1(\mathbb{R})$

Recall: $f, g \in C_m(\mathbb{R})$ (or $L^1(\mathbb{R})$), then

i) $\hat{f}(\xi)$ well-def.

ii) $\hat{f} \in C(\mathbb{R})$, $\hat{f} \underset{|\xi|\to\infty}{\longrightarrow} 0$, $\|\hat{f}\|_\infty \le \|f\|_1$

iii) $f * g \in C_m(\mathbb{R})$ ($\in L^1(\mathbb{R})$) [Exercise 1]

iv) $\mathcal{F}[e^{-2\pi i x h} f(x)](\xi) = \hat{f}(x+h)$ v) $\mathcal{F}^{-1}[f(x)](\xi) = \mathcal{F}[f](-\xi)$

Prop. 1:

(c) $f, g \in C_m(\mathbb{R}) \Rightarrow \int_{\mathbb{R}} f \cdot \hat{g} = \int_{\mathbb{R}} \hat{f} \cdot g$

(b) $f, g \in C_m(\mathbb{R}) \Rightarrow \widehat{f * g} = \hat{f} \cdot \hat{g}$

Thm. 1: $f, \hat{f} \in C_m(\mathbb{R}) \Rightarrow f(x) = \mathcal{F}^{-1}[\hat{f}](x)$

(a) $f, f' \in C_m(\mathbb{R}) \Rightarrow \widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$

Pf's: As in $S(\mathbb{R})$ – same arguments (chk) □

Rem. 1:

(a) Can replace C_m by L^1 in Prop. 1 and Thm. 1

(b) $f \in C_m$ (or L^1) $\not\Rightarrow \hat{f} \in C_m$ (or L^1) [see Ex. 1]

~~(c) "Riemann-Lebesgue": $\hat{f} \underset{|\xi|\to 0}{\longrightarrow} 0$ (see ii))~~

Rem. 2: $\underbrace{\cancel{\text{Inversion when } f, \hat{f} \in C_m}}$

When is $\hat{f} \in C_m$?

Ex.: $f, f', f'' \in C_m$ (or L^1)

$$\Rightarrow \hat{f} \in C_m$$

$$\Big[\ |\hat{f}(\xi)| \overset{\text{Prop. 1a}}{=} \Big| -\frac{1}{(2\pi\xi)^2} \widehat{f''}(\xi) \Big| \overset{ii)}{\leq} \frac{1}{4\pi^2\xi^2} \|f''\|_1$$

$$|\hat{f}(\xi)| \overset{ii)}{\leq} \|f\|_{L^1}$$

$$\Rightarrow |\hat{f}(\xi)| \leq \frac{2\|f\|_1 + \|f''\|_1}{1 + \xi^2} \ \Big]$$

<u>Ex. 1:</u> $\chi_{(-a,a)}(x) = \begin{cases} 1 , & |x|<a \\ 0 , & |x| \geq a \end{cases} \quad , \ a>0$

$$\widehat{\chi_{(-a,a)}}(\xi) = \int_{-a}^{a} e^{-2\pi i x \xi}\,dx = -\frac{1}{2\pi i \xi}\left(e^{-2\pi i a \xi} - e^{+2\pi i a \xi}\right)$$

$$= \frac{1}{\pi} \cdot \frac{\sin a\xi}{\xi}$$

Obs: $\chi_{(-a,a)} \in PC_m \subseteq L^1$, $\widehat{\chi}_{(-a,a)} \notin L^1$ $\left[|\widehat{\chi}_a| \sim \frac{1}{\pi} \frac{1}{|\xi|} \right]$

<u>Ex. 2:</u> $K_\delta(x) = \delta^{-\frac{1}{2}} K(\delta^{-\frac{1}{2}} x)$, $K(x) = e^{-\pi x^2}$

Prev. : $\widehat{K}_\delta(\xi) = e^{-\pi \delta \xi^2}$

$K_\delta, \widehat{K}_\delta \in S(\mathbb{R}) \subset C_m(\mathbb{R})$

<u>Ex. 3:</u> $f(x) = \dfrac{1}{x^2+a^2}$, $a>0$

Chk: $\widehat{f}(\xi) = \frac{\pi}{a} e^{-2\pi a |\xi|}$ [Exercise 8]

Obs: $f, \widehat{f} \in C_m(\mathbb{R}) \subset L^1(\mathbb{R})$

<u>Lem 1:</u> Riemann-Lebesgue

$$\underbrace{f \in PC_m(\mathbb{R})}_{(\text{or } L^1(\mathbb{R}))} \Rightarrow \widehat{f}(\xi) \xrightarrow[|\xi| \to \infty]{} 0$$

<u>Pf.:</u>

Take $f_n \in C_m$ s.t. $\| f_n - f \|_1 < \frac{\varepsilon}{2}$ [~~must be proved~~, Exercise 8]

Take N s.t. $\underbrace{|\widehat{f}_n(\xi)| < \frac{\varepsilon}{2}}_{\text{for } |\xi|>N}$ (by ii)).

Then for $|\xi| > N$,

$$|\widehat{f}(\xi)| \leq |\widehat{f}_n(\xi)| + \| \widehat{f} - \widehat{f}_n \|_\infty$$

$$\overset{ii)}{\leq} \frac{\varepsilon}{2} + \| f - f_n \|_1 < \varepsilon \qquad \square$$

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How to recover $f \in C_m\,(L^1)$ from $\hat{f}$ when $\hat{f} \notin C_m\,(L^1)$?

A Fejer type of result:

<u>Thm. 2:</u> Let $\hat{K}_\delta(\xi) = e^{-\pi\delta\xi^2}$

$$f \in C_m \text{ (or } L^1) \Rightarrow f(x) = \lim_{\delta \to 0} \mathcal{F}^{-1}[\hat{K}_\delta(\xi)\hat{f}(\xi)](x)$$

~~Recall:~~ $\cancel{K_\delta(x) = \delta^{-\frac{1}{2}}K(\delta^{-\frac{1}{2}}x),\ K(x) = e^{-\pi x^2},\ \hat{K}_\delta(\xi) = e^{-\pi\delta\xi^2}}$.

<u>Obs. 1:</u>

(a) $|\hat{K}_\delta \hat{f}| \le e^{-\pi\delta\xi^2}\|f\|_1 \overset{\text{iii)}}{\Rightarrow} \hat{K}_\delta \hat{f} \in C_m$ (or L^1)

$\Rightarrow \mathcal{F}^{-1}[\hat{K}_\delta \hat{f}]$ well-def.

(b) $\hat{K}_\delta \cdot \hat{f} \overset{\text{Prop. 1b}}{=} \widehat{K_\delta * f}$, $K_\delta * f \overset{\text{iii)}}{\in} C_m$ (or L^1)

$\overset{\text{Thm. 1}}{\Longrightarrow} K_\delta * f\,(x) = \mathcal{F}^{-1}[\hat{K}_\delta \hat{f}](x)$ $\forall x$ (or in L^1)

(c) K_δ is a good kernel, so

$K_\delta * f(x) \longrightarrow f(x)$ unif. x (or in $L^1(\mathbb{R})$)

<u>Pf.:</u> Obs. 1 (a) – (c) when $f \in C_m(\mathbb{R})$. □

(Also works for $f \in L^1$) □

Pointwise conv. where f is more regular:

<u>Thm. 3:</u> Assume $f \in \overset{\text{p.w. cont.}}{PC_m}(\mathbb{R})$ and $D^{\pm}f(x)$ exist.

Then $\lim\limits_{R\to\infty} \mathcal{F}^{-1}_R[\hat{f}](x) = \frac{1}{2}(f(x^+) + f(x^-))$.

<u>Rem. 3:</u> Can replace $f \in PC_m$ by $f \in L^1$ and $f(x^{\pm})$ exist. 9:00

Lem 1

p. 2

——— Lem. 1 ———

<u>Pf. of Thm. 3</u>

1) $\int_{-R}^{R} e^{+2\pi i x \xi}\, d\xi = \frac{1}{+2\pi i x}\left(e^{+2\pi i x R} - e^{-2\pi i x R}\right) = \frac{1}{\pi x} \sin 2\pi x R = \tilde{D}_R(x)$

2) $\mathcal{F}^{-1}[\hat{f}](x) = \lim\limits_{R\to\infty} I_R$,

$I_R = \int_{-R}^{R} \underbrace{\hat{f}(\xi)\, e^{+2\pi i x \xi}}\, d\xi$

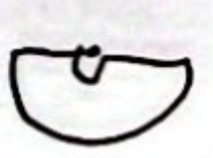
$= \int_{-R}^{R} \int_{\mathbb{R}} f(y)\, e^{2\pi i \xi(x-y)}\, dy\, d\xi$

$\overset{\text{interchange int's}}{=} \int_{\mathbb{R}} f(y) \int_{-R}^{R} e^{2\pi i \xi(x-y)}\, d\xi\, dy$

$\overset{1)}{=} \int_{\mathbb{R}} f(y)\, \tilde{D}_R(x-y)\, dy = \int_{\mathbb{R}} f(x-y)\, \tilde{D}_R(y)\, dy$

3) $\int_{\mathbb{R}} |\tilde{D}_R| \sim \int_{\mathbb{R}} \frac{1}{|x|} = \infty$ (not good kernel)

But : $\int_0^\infty \tilde{D}_R = \int_{-\infty}^{0} \tilde{D}_R = \frac{1}{2}$

[Chk.: Residue thm. $\oint_{D_{R,r}} f$, $f = \frac{e^{2\pi i R z}}{\pi z}$, $D_{R,r} =$ ]

4) $\underbrace{I_R - \frac{1}{2}(f(x^-) + f(x^+))}_{\Delta_R}$

3)

$$= \int_{-\infty}^{0} (f(x-y) - f(x^+))\, \tilde{D}_R(y)\, dy$$

$$+ \int_0^{\infty} (f(x-y) - f(x^-))\, \tilde{D}_R(y)\, dy = A_R + B_R$$

$$= \int_{\mathbb{R}} g(y) \cdot \sin(2\pi y R)\, dy ,$$

where

$$g(y) = \begin{cases} \dfrac{f(x-y) - f(x^+)}{y} , & y < 0 \\[2ex] \dfrac{f(x-y) - f(x^-)}{y} , & y > 0 \end{cases}$$

5) $$|B_R| \overset{\text{chk}}{\leq} \underbrace{\int_K^{\infty} f(x-y)\, \tilde{D}_R(y)\, dy}_{\substack{\leq \int_K^{\infty} |f(x-y)|\, dy \\ \downarrow K\to\infty \\ 0 \quad f \in PC_m}} - \underbrace{f(x^-) \int_K^{\infty} \tilde{D}_R(y)\, dy}_{\substack{K\to\infty \downarrow \ 3) \\ 0}} + \underbrace{\int_0^K (\cdots)\, dy}_{= \tilde{B}_{R,K}} +$$

$$\tilde{B}_{R,K} = \int_{\mathbb{R}} g_K(y) \left(e^{2\pi i y R} - e^{-2\pi i y R}\right) dy = \hat{g}_K(-R) - \hat{g}_K(R)$$

where

$$g_K(y) = \begin{cases} \dfrac{1}{2i} \dfrac{f(x-y) - f(x^-)}{y} , & y \in (0, K) \\[2ex] 0 , & \text{otherwise} \end{cases}$$

Obs: $\left.\begin{array}{l} g_K \in \text{p.w. cont. for } y \neq 0 \\ \lim\limits_{y\to 0^+} g_K(y) = -\dfrac{1}{2\pi i} D^- f(x) \end{array}\right\} \Rightarrow g_K$ p.w. cont.

Hence $g \in PC_m(\mathbb{R})$ $\quad [g = 0 \text{ for } |y| > K]$

and by "R.-L." (Lem 1), $\hat{g}_K(\pm R) \xrightarrow[R\to\infty]{} 0$

For $\varepsilon > 0$, take K large s.t.

$$\int_K^\infty (\cdots) < \frac{\varepsilon}{2}$$

and then R large so

$$\int_0^K (\cdots) < \frac{\varepsilon}{2}$$

Hence $|B_R| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$.

So $B_R \xrightarrow[R\to\infty]{} 0$. Similarly $A_R \xrightarrow[R\to\infty]{} 0$, and

$$\Delta_R = A_R + B_R \xrightarrow[R\to\infty]{} 0 \qquad \square$$

F. anal. 6. 3. 2025

- Slides, info (ref q-p note)

A. Fourier transform in L^2

Recall: $f, g \in S(\mathbb{R})$

i) $\hat{f} \in S(\mathbb{R})$

ii) $\mathcal{F}^{-1}[\hat{f}](x) = f(x)$

iii) $\widehat{f * g} = \hat{f} \cdot \hat{g}$

iv) $\underbrace{\|f\|_2 = \|\hat{f}\|_2}_{\text{(Plancherel)}}$, $\int_{\mathbb{R}} f \bar{g} = \int_{\mathbb{R}} \hat{f} \bar{\hat{g}}$

More general theory in L^1, but *no* Plancherel.

Plancherel is an L^2 result!

Ex. 1: $L^1(\mathbb{R}) \not\subset L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$

$g = x^{-\frac{1}{2}} \chi_{[0,1]}(x) \in L^1$, but $\notin L^2$ $[\|g\|_2 = \infty]$

$h = \frac{1}{(1+|x|^2)^{\frac{1}{2}}}$, $h^2 \in C_m \subset L^1 \Rightarrow h \in L^2$; but $\notin L^1$ $[\|h\|_1 = \infty]$

Rem. 1:

a) $|f|^2 \in L^1 \Leftrightarrow f \in L^2$

b) $g, |h|^2 \in C_m \Rightarrow g \in L^1$; $g, h \in L^2$ (chk)

Lem. 1: $f \in L^2(\mathbb{R}) \Rightarrow \exists \{f_n\}_n \subset S(\mathbb{R})$ s.t. $\|f_n - f\|_2 \to 0$

Pf.:

1) Take $N = N(n)$ s.t. $\int_{|x|>N} |f|^2 < \frac{1}{n}$ ("dominated conv.", from Lebesgue theory)

2) Prev. results: (not proved, L.-theory)

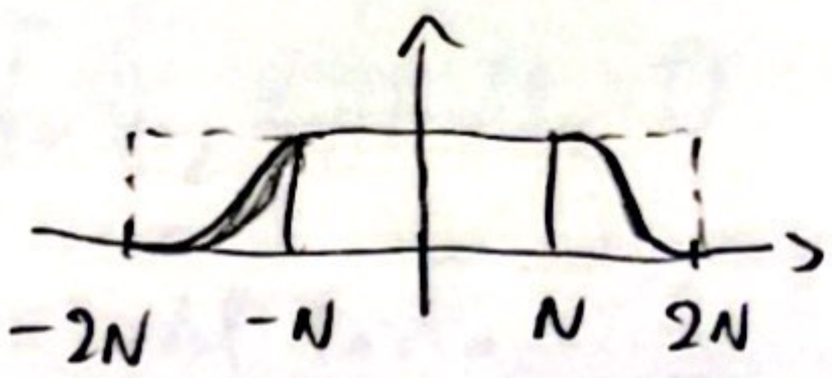
$\exists \tilde{f}_n \in C([-2N, 2N])$ s.t. $\int_{|x|<2N} |f - \tilde{f}_n|^2 < \frac{1}{n}$

3) Weierstrass approx. thm.:

$\exists$ polynomial p_n s.t. $\int_{|x|<2N} |\tilde{f}_n - p_n|^2 \leq 4N \max_{|x|\leq 2N} |\tilde{f}_n - p_n|^2 < \frac{1}{n}$

4) Let $\psi_N \in C^\infty(\mathbb{R})$ be a bump f'n s.t.

$$\chi_{[-N,N]}(x) \le \psi_N(x) \le \chi_{[-2N,2N]}(x)$$



$$f_n(x) := \psi_N(x) \cdot p_n(x) \in C^\infty, \ = 0 \text{ for } |x| > 2N$$

$$\Rightarrow f_n \in S(\mathbb{R}) \quad (\in C_c^\infty(\mathbb{R}))$$

Obs: $f_n = p_n$ for $|x| < N$

5) $$\int_{|x|>N} |f_n|^2 \overset{\psi_N = 0,\, |x|>2N}{\le} \int_{N<|x|<2N} |p_n|^2 \le \int_{N<|x|<2N} 3(|f|^2 + |f-\tilde{f}_n|^2 + |\tilde{f}_n - p_n|^2$$

$|\psi_N| \le 1$

$$\le 3 \quad |p_n| \le |f| + |f-\tilde{f}_n| + |\tilde{f}_n - p_n| \quad < 3\left(\frac{1}{n} + \frac{1}{n} + \frac{1}{n}\right)$$

by 1)-3)

6) Conclusion:

$$\int_{\mathbb{R}} |f - f_n| \le \int_{|x|>R} \underbrace{(|f| + |f_n|)^2}_{= |f|^2 + \underbrace{2|f||f_n|}_{\le |f|^2 + |f_n|^2} + |f_n|^2} + \int_{|x|<R} (|f - \tilde{f}_n| + |\tilde{f}_n - p_n| + \underbrace{|p_n - f_n|}_{=0})^2$$

$$\le 2\int_{|x|>N}(|f|^2 + |f_n|^2) + 2\int_{|x|<N}(|f - \tilde{f}_n|^2 + |\tilde{f}_n - p_n|^2)$$

11:00

$$\overset{1)-5)}{<} 2\left(\frac{1}{n} + \frac{9}{n}\right) + 2\left(\frac{1}{n} + \frac{1}{n}\right) \underset{n\to\infty}{\longrightarrow} 0 \qquad \square$$

Obs. 1: Let $f \in L^2(\mathbb{R})$

i) By Lem 1, $\exists f_n \in S(\mathbb{R})$ s.t. $\|f - f_n\|_2 \to 0$.

ii) $\overset{i)}{\hat{f}_n \in S(\mathbb{R})}$ and $\{\hat{f}_n\}_n$ Cauchy in $L^2(\mathbb{R})$:

$$\|\hat{f}_n - \hat{f}_m\|_2 \underset{iv)}{\overset{\text{Plancherel}}{=}} \|f_n - f_m\|_2 \underset{n,m\to\infty}{\longrightarrow} 0$$

iii) Since L^2 complete, $\exists g \in L^2$ s.t. $\hat{f}_n \xrightarrow{L^2} g$

iv) g only dep. on f and not on $\{f_n\}_n$:

If $f_n, \tilde{f}_n \to f$ and $\hat{f}_n, \hat{\tilde{f}}_n \to g, \tilde{g}$, then $g = \tilde{g}$:

$$\|g - \tilde{g}\|_2 \underset{\substack{\uparrow \\ \text{def. } g, \tilde{g}}}{=} \lim \|\hat{f}_n - \hat{\tilde{f}}_n\| \underset{\substack{\uparrow \\ \text{Plancherel}}}{=} \lim \|f_n - \tilde{f}_n\|_2 \underset{\substack{\uparrow \\ \text{def. } f_n, \tilde{f}_n}}{=} \|f - f\|_2 = 0$$

<u>Def. 1:</u> $f \in L^2(\mathbb{R})$, $\hat{f} := L^2\text{-}\lim \hat{f}_n$

for any $\{f_n\}_n \subset S(\mathbb{R})$ s.t. $f_n \xrightarrow{L^2} f$.

<u>Rem. 2:</u>

a) $f \in L^1 \cap L^2$, the 2 def's give same $\hat{f}$

b) $f \in L^2$ and $\notin L^1$; then (e.g.)

$$\hat{f}(\xi) = L^2\text{-}\lim_{\delta \to 0^+} \int_{\mathbb{R}} f(x) e^{-\pi \delta x^2} e^{-2\pi i x \xi} \, dx$$

c) "=" here means in L^2 and "almost every point" (a.e.)

["$f = g$" $\Leftrightarrow \|f - g\|_2 = 0 \Leftrightarrow f(x) = g(x), x \in N^c$, $N \subset \mathbb{R}$ has measure 0]

<u>Thm. 1:</u> The $\mathcal{F}$-transform, def. on $S(\mathbb{R})$ (and $L^1(\mathbb{R})$), extends uniquely to $L^2(\mathbb{R})$ and satisfies

i) $\|f\|_2 = \|\hat{f}\|_2$, $(f, g)_2 = (\hat{f}, \hat{g})_2$

ii) $f(x) = \mathcal{F}^{-1}[\hat{f}](x)$ (in L^2 and a.e.)

iii) $\mathcal{F}^{-1}[\hat{f} \cdot \hat{g}](x) = f * g(x)$, $x \in \mathbb{R}$

iv) All other properties of $\mathcal{F}$ also holds (assumptions on f' ...)

(iv) $\mathcal{F}^{-1}[g(\xi)](x) = \mathcal{F}[g(-\xi)](x) = \mathcal{F}[g](-x)$ (—"—)

Rem. 3:

a) $f, g \in L^2 \Rightarrow$ (i) $\|f*g\|_\infty \overset{C-S}{\leq} \|f\|_2 \|g\|_2$, $f*g$ well-def.

(ii) $\hat{f}\cdot\hat{g} \in L^1$, so $F^{-1}[\hat{f}\cdot\hat{g}]$ well-def.

b) By i) and ii), $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

is invertible and preserve $\|\cdot\|_2$ (isometry)

Pf. :

i) $\|f\|_2 = \lim \|f_n\|_2 \overset{\text{Plancherel in } S(\mathbb{R})}{=} \lim \|\hat{f}_n\|_2 = \|\hat{f}\|_2$

ii) $\|f - F^{-1}[\hat{f}]\|_2$

$$\leq \|f - f_n\|_2 + \underbrace{\|f_n - F^{-1}[\hat{f}_n]\|_2}_{\substack{=0 \\ \text{ii) inversion in } S(\mathbb{R})}} + \underbrace{\|F^{-1}[\hat{f}_n - \hat{f}]\|_2}_{= \|\hat{f}_n - \hat{f}\|_2 = \|f_n - f\|_2}$$

Plancherel in L^2

$\xrightarrow[n\to\infty]{\text{send}} 0$

iii) Let $h(y) = \overline{g(x-y)}$, then $\hat{h}(\xi) = \overline{\hat{g}(\xi)}\, e^{-2\pi i \xi \cdot x}$

$f*g(x) = \int f\bar{h} \overset{\text{Plancherel}}{=} \int \hat{f}\cdot\bar{\hat{h}} = \int \hat{f}\cdot\hat{g}\, e^{2\pi i \xi\cdot x} = F^{-1}[\hat{f}\cdot\hat{g}](x)$

Etc. $\square$

B. Computing $\hat{f}$ via $\mathcal{F}$-inversion

Assume $f = \hat{\varphi}$ and $\varphi \in L^2(\mathbb{R})$ is known:

$$\varphi(\xi) \overset{F\text{-inv.}}{=} \mathcal{F}^{-1}[f](\xi) \overset{\text{Thm. 1 iv}}{=} \hat{f}(-\xi)$$

$$\Downarrow \text{Thm. 1 (ii)}$$

$$\Rightarrow \quad f(x) = \varphi(-x)$$

(1) $\boxed{\hat{f}(\xi) = \varphi(-\xi)}$

Ex. 2: a) $f_1(x) = \frac{1}{1+x^2} \in C_m \subset L^2$

$$f_1(x) \overset{\text{prev.}}{=} \mathcal{F}[\pi \underbrace{e^{-2\pi|\xi|}}_{\in L^2}](x), \quad e^{-2\pi|\xi|} \in C_m \subset L^2.$$

$$\overset{(1)}{\Rightarrow} \hat{f}_1(\xi) = \pi e^{-2\pi|\xi|}$$

b) $f_2(x) = \frac{\sin x}{x} \in C \cap L^2, \notin L^1$ (chk!)

$$f_2(x) \overset{\text{prev.}}{=} \mathcal{F}[\pi \chi_{[-1,1]}(\xi)](x)$$

$$\overset{(1)}{\Longrightarrow} \hat{f}_2(\xi) = \pi \chi_{[-1,1]}(-\xi) = \pi \chi_{[-1,1]}(\xi)$$

Ex. 3: Using other rules

a) $f_3(x) = \frac{2x}{(1+x^2)^2} \overset{\text{chk}}{=} -f_1'(x)$

$$\Rightarrow \hat{f}_3(\xi) = -\widehat{f_1'}(\xi) = -2\pi i \xi \hat{f}_1(\xi) = -2\pi^2 i \xi e^{-2\pi|\xi|}$$

b) $\mathcal{F}[\chi_{[x_0-1, x_0+1]}(x)] = \mathcal{F}[\chi_{[-1,1]}(x-x_0)] = e^{-2\pi i x_0} \frac{1}{\pi} \frac{\sin x}{x}$

Legg inn oppg. i rø v. 9.